Summary of Angular Momentum

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We start from the classical expression for angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p}$, to obtain the quantum mechanical version $\hat{\mathbf{L}} = \hat{\mathbf{R}} \times \hat{\mathbf{P}}$, where $\hat{\mathbf{L}}$, $\hat{\mathbf{R}}$, and $\hat{\mathbf{P}}$ are all three-dimensional vectors. This definition leads immediately to expressions for the three components of $\mathbf{L}$:

$$\hat{L}_x = \hat{Y}\hat{P}_z - \hat{Z}\hat{P}_y \tag{1}$$
$$\hat{L}_y = \hat{Z}\hat{P}_x - \hat{X}\hat{P}_z \tag{2}$$
$$\hat{L}_z = \hat{X}\hat{P}_y - \hat{Y}\hat{P}_x. \tag{3}$$

From these definitions, we may easily derive the following commutators

$$\left[\hat{L}_i, \hat{R}_j\right] = \epsilon_{ijk} i\hbar \hat{R}_k \tag{4}$$
$$\left[\hat{L}_i, \hat{P}_j\right] = \epsilon_{ijk} i\hbar \hat{P}_k \tag{5}$$
$$\left[\hat{L}_i, \hat{L}_j\right] = \epsilon_{ijk} i\hbar \hat{L}_k, \tag{6}$$

where the indices i, j, k can be x, y, or z, and where the coefficient ϵ_{ijk} is unity if i, j, k form a cyclic permutation of x, y, z [i.e., (x, y, z), (y, z, x), or (z, x, y)] and -1 for a reverse cyclic permutation [(z, y, x), (x, z, y), or (y, x, z)]. The final commutator indicates that we cannot generally know L_x, L_y, and L_z simultaneously except if we have an eigenstate with eigenvalue 0 for each of these.

Classically, any component of the angular momentum must be less than or equal to the magnitude of the overall angular momentum vector. Quantum mechanically, the *average* value of any component of the angular momentum must be less than or equal to the square root of the expectation value of $\hat{\mathbf{L}}$ dotted with itself:

$$< \hat{L}_i > \leq \sqrt{< \hat{\mathbf{L}}^2 >}. \tag{7}$$

$\hat{\mathbf{L}}^2$ is simply $\hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$.

Since $\hat{\mathbf{L}}^2$ commutes with any component $\hat{L}_i$, we *can* have simultaneous eigenfunctions of $\hat{\mathbf{L}}^2$ and a given component $\hat{L}_i$. We usually pick the z axis, since the expression for $\hat{L}_z$ is the easiest of the three when we work in spherical polar coordinates:

$$\hat{L}_x = \frac{\hbar}{i}\left(-sin\phi \frac{\partial}{\partial\theta} - cos\phi \, cot\theta \frac{\partial}{\partial\phi}\right) \tag{8}$$

$$\hat{L}_y = \frac{\hbar}{i}\left(cos\phi\,\frac{\partial}{\partial\theta} - sin\phi\,cot\theta\,\frac{\partial}{\partial\phi}\right) \tag{9}$$

$$\hat{L}_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}. \tag{10}$$

Of course it is also possible to express $\hat{\mathbf{L}}$ in terms of the unit vectors for spherical polar coordinates,

$$\hat{r} = sin\theta\,cos\phi\,\hat{i} + sin\theta\,sin\phi\,\hat{j} + cos\theta\,\hat{k} \tag{11}$$

$$\hat{\phi} = -sin\phi\,\hat{i} + cos\phi\,\hat{j} \tag{12}$$

$$\hat{\theta} = cos\theta\,cos\phi\,\hat{i} + cos\theta\,sin\phi\,\hat{j} - sin\theta\,\hat{k}. \tag{13}$$

Here,

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{rsin\theta}\frac{\partial}{\partial\phi} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} \tag{14}$$

and

$$\begin{aligned} \hat{\mathbf{L}} &= \hat{\mathbf{R}} \times \hat{\mathbf{P}} \\ &= r\hat{r} \times \frac{\hbar}{i}\nabla \\ &= \frac{\hbar}{i}\left(\hat{\phi}\frac{\partial}{\partial\theta} - \hat{\theta}\frac{1}{sin\theta}\frac{\partial}{\partial\phi}\right). \end{aligned} \tag{15}$$

The simultaneous eigenfunctions of $\hat{\mathbf{L}}^2$ and $\hat{L}_z$ are called the *spherical harmonics*, $Y_l^m(\theta,\phi)$, where l is the total angular momentum quantum number, and m is the so-called magnetic quantum number. The spherical harmonics are defined as

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(-1)^m e^{im\phi} P_l^m(cos\theta), \tag{16}$$

where P_l^m are the *associated Legendre polynomials*. We require that $-l \leq m \leq l$, and spherical harmonics with $m < 0$ are defined in terms of the spherical harmonics with $m > 0$ according to $Y_l^m = (-1)^m[Y_l^{-m}]^*$. The spherical harmonics are normalized over integration of angular coordinates such that

$$\int_0^\pi sin\theta\,d\theta \int_0^{2\pi} d\phi\,[Y_{l'}^{m'}(\theta,\phi)]^* Y_l^m(\theta,\phi) = \delta_{l,l'}\delta_{m,m'}, \tag{17}$$

and they have the following special properties:

$$\hat{L}_z Y_l^m = m\hbar Y_l^m \tag{18}$$

$$\hat{\mathbf{L}}^2 Y_l^m = l(l+1)\hbar^2 Y_l^m. \tag{19}$$

It can be useful to define *ladder operators* for angular momentum. The following ladder operators work not only for straight angular momentum $\hat{\mathbf{L}}$, but also for combined angular momenta such as $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \hat{\mathbf{S}}$. If

$$\hat{\mathbf{J}}_{\pm} = \hat{J}_x \pm i\hat{J}_y \tag{20}$$

then

$$\hat{\mathbf{J}}_{\pm}|jm\rangle = \hbar\left[(j \mp m)(j \pm m + 1)\right]^{1/2}|j, m \pm 1\rangle. \tag{21}$$

We can see that these ladder operators raise or lower the magnetic quantum number m but leave l alone.

One can also show that in spherical polar coordinates

$$\hat{\mathbf{L}}^2 = -\hbar^2\left[\frac{1}{sin^2\theta}\frac{\partial^2}{\partial\phi^2} + \frac{1}{sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right)\right]. \tag{22}$$

By comparing this expression with that for ∇^2 in spherical polar coordinates,

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial\theta}\left(sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2 sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2}\right), \tag{23}$$

we can see that the Hamiltonian can be written as

$$\begin{aligned}\hat{H} &= -\frac{\hbar^2}{2m}\nabla^2 + \hat{V} \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{\hat{\mathbf{L}}^2}{2mr^2} + \hat{V}.\end{aligned} \tag{24}$$

Clearly $\hat{\mathbf{L}}^2$ commutes with the kinetic energy term, $\hat{\mathbf{L}}^2$ has no r dependence. Likewise, if $\hat{V} = V(r)$, then $\hat{\mathbf{L}}^2$ commutes with the whole Hamiltonian. Hence, for problems where the potential depends only on r (central force problems), we can find simultaneous eigenfunctions of $\hat{H}$, $\hat{\mathbf{L}}^2$, and $\hat{L}_z$.