Notes on Elementary Linear Algebra

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1 Vectors

In quantum mechanics, we are generally interested in complex numbers. Let $\mathbf{C}^n$ denote the set of all n-tuples of complex numbers (a *complex n-space*). The elements of $\mathbf{C}^n$ may be called "points" or "vectors" in complex n-space. The elements of the complex numbers $\mathbf{C}$ are *scalars*.

A vector might be denoted by listing all its elements as $(c_1 c_2 c_3 \cdots c_n)$. This format is called a *row vector*. Alternatively, the elements could be arranged as

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{pmatrix},$$

which is a *column vector*. Our default choice will be to write vectors as column vectors, and row vectors will result from taking (complex conjugate) transposes of vectors.

For real numbers, the dot (or inner) product between two vectors $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1 a_2 \cdots a_n) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= a_1 b_1 + a_2 b_2 + \cdots + a_n b_n \end{aligned} \tag{1}$$

If $\mathbf{a}$ and $\mathbf{b}$ are complex, we have to take the complex transpose of $\mathbf{a}$,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_1^* a_2^* \cdots a_n^*) \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} \\ &= a_1^* b_1 + a_2^* b_2 + \cdots + a_n^* b_n. \end{aligned} \tag{2}$$

Recall that the complex conjugate of a complex number like $c + id$ is $c - id$ (the imaginary part has its sign reversed).

2 Matrices

A matrix is a rectangular collection of numbers (again, possibly complex). It might represent a collection of row/column vectors, or a transformation that changes a vector into a different vector. For example, the rotation of a real 2D vector by an angle θ about the z axis could be represented as a 2x2 matrix,

$$\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}.$$

The effect of this rotation on a 2D vector in the x, y plane can be determined by multiplying the vector by this transformation matrix,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (3)$$

where $x' = xcos\theta - ysin\theta$ and $y' = xsin\theta + ycos\theta$.

The *transpose* of a matrix $\mathbf{A}$, typically denoted $\mathbf{A}^T$, is obtained simply by swapping elements across the diagonal, $a_{ij} \rightarrow a_{ji}$. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}. \quad (4)$$

For complex numbers, usually the *complex conjugate transpose*, or *adjoint*, is more useful. The adjoint of a matrix $\mathbf{A}$ is often denoted $\mathbf{A}^\dagger$. For example,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} a^* & c^* \\ b^* & d^* \end{bmatrix}. \quad (5)$$

If a matrix $\mathbf{A}$ is equal to its adjoint $\mathbf{A}^\dagger$, it is said to be a *Hermitian* matrix. Clearly, this can only happen if the diagonal elements are real (otherwise, a will never equal a^*, for example).

3 Determinants

The determinant of a matrix $\mathbf{A}$ is denoted $|\mathbf{A}|$ and is a scalar quantity (i.e., a number). This number is involved in computation of inverse matrices (below). For the trivial case of a 1x1 matrix, the determinant is just the number in the matrix. For a 2x2 matrix, the determinant is easily computed as

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = A_{11}A_{22} - A_{21}A_{12}. \tag{6}$$

For a diagonal (or triangular) matrix, the determinant is just the product of the entries on the main diagonal.

For larger or more general matrices, there is a procedure for computing determinants which involves cofactor matrices

$$\text{det}\mathbf{A} = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}, \tag{7}$$

where a_{ij} are the elements of matrix A and A_{ij} are the *cofactors*, which are $(-1)^{i+j}$ times the determinant of a smaller $(n-1) \times (n-1)$ matrix found by eliminating row i and column j $[A_{ij} = (-1)^{i+j}\text{det}\mathbf{M}_{ij}]$. Obviously there is a problem if we are defining a determinant in terms of other determinants! However, we can apply these rules iteratively until we get to 2x2 matrices, for which we can take determinants using the simple rule given above.

4 Inverse Matrices

The *inverse* of a matrix is another matrix which, when multiplied by the first matrix, yields the unit matrix **I** (a matrix with all zeroes except 1's down the diagonal).

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}. \tag{8}$$

In the general case, the inverse may be written

$$\mathbf{A}^{-1} = \frac{1}{\text{det}\mathbf{A}}\mathbf{A}^T_{cof}, \tag{9}$$

where $\mathbf{A}^T_{cof}$ is the transpose of the matrix of cofactors $A_{ij} = (-1)^{i+j}det\mathbf{M}_{ij}$. Example:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ab - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{10}$$

Clearly, there are major problems in finding the inverse of a matrix if it has a determinant equal to zero, since the formula for $\mathbf{A}^{-1}$ involves dividing by the determinant. In such cases, we say that $\mathbf{A}$ is *singular* and has no inverse.

5 Eigenvectors and Eigenvalues

The eigenvectors of a matrix $\mathbf{A}$ are those special vectors $\mathbf{v}$ for which $\mathbf{Av} = \lambda\mathbf{v}$, where λ is an associated constant (possibly complex) called the eigenvalue. Various methods exist for finding eigenvalues. Generally, one solves for $(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0$. This occurs for special values of λ which can be determined by finding those λ for which the *secular determinant* $|\mathbf{A} - \lambda\mathbf{I}| = 0$.